

# Black Hole Entropy in Loop Quantum Gravity and Number Theory

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## Abstract

We show that counting different configurations that give rise to black hole entropy in loop quantum gravity is related to partitions in number theory.

## 1 Introduction

The microscopic description of the Bekenstein-Hawking entropy [1], [2] is one of the most important problems any theory of quantum gravity should explain.

In loop quantum gravity black hole entropy has been studied well for isolated horizons and of large area. One of the most fundamental problems for completing the task is to know exactly how many different configurations we have that give rise to a fixed area.

There is an effective method for counting configurations by using methods of number theory which was studied in [3].

In this paper we show that there is in fact a deeper relation between the counting of configurations which give rise to black hole entropy and certain counting in number theory.

We start our solution by considering first the simplest case which is given when the area spectrum is equidistant. This situation for example emerges naturally in loop quantum gravity when we are considering very large spins.

In loop quantum gravity spin network states are eigenvalues of the area operator. The spin network edges are labelled by half-integers  $\{j \in 0, 1/2, 1, \dots\}$ . When a surface is punctured by an edge labelled with a spin  $j$  the surface

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acquires the area  $A_j = 8\pi\gamma l_p^2 \sqrt{j(j+1)}$ , where  $l_p$  is the Planck length and  $\gamma$  is a parameter known by the name of Immirzi.

More specifically, label the edges of the spin network by  $j_i$  which are half integers, that is, irreducible representations of the group  $SU(2)$ . Suppose the spin network punctures the surface in  $n$  isolated points, and in a non-degenerate way. Consider units for which  $4\pi\gamma l_p^2 = 1$ . The total area of the surface is given by the eigenvalues of the area operator

$$A = 2 \sum_{i=1}^n \sqrt{j_i(j_i + 1)} \quad (1)$$

Now, if we have an isolated black hole, the microscopic description of its entropy is given by states which live in the horizon surface. This entropy is given by the logarithm of the number  $\mathcal{N}$  of microstates which account for a fixed area of the surface.

The number of microstates is given as follows. It is known that each spin network edge labelled  $j$  which punctures the horizon contributes to the dimension of the boundary Hilbert space of states by a factor  $(2j+1)$  which is the dimension of the irreducible representation  $j$  of  $SU(2)$ . When considering all the punctures of the horizon the dimension of the Hilbert space is given by the product of all the numbers  $(2j+1)$  associated to the spins which label the puncturing edges.

The entropy is given by the logarithm of the dimension of the Hilbert space of the boundary.

$$S = \ln \mathcal{N} \quad (2)$$

The problem we consider here is the counting of configurations which account for a fixed area of the black hole horizon. We also should take into account whether we are considering counting distinguishable or indistinguishable configurations.

A configuration is a set of edges of a spin network puncturing the horizon in a non-degenerate way and labelled  $\{n_j\} = \{n_{1/2}, n_1, \dots, n_{s_{max}/2}\}$  where  $n_j$  is the number of punctures with spin  $j$ , and where the following equation is satisfied

$$A = 2[n_{1/2}\sqrt{j_{1/2}(j_{1/2} + 1)} + n_1\sqrt{j_1(j_1 + 1)} + \dots + n_{s_{max}/2}\sqrt{j_{s_{max}/2}(j_{s_{max}/2} + 1)}] \quad (3)$$

We can ask equation (3) to be satisfied exactly or we can also ask for configurations which area eigenvalue lies in an interval  $[A - \delta, A + \delta]$ . How many configurations satisfy equation (3)?

The problem of counting the number of these configurations started interestingly with ideas of [4], [5]. Then it also has been considered for example in [3], [6], [7], [8], [9], [10]; however the states which account for the entropy vary in opinions and we have various possibilities.

For example, it has also been discussed whether two states which may vary by a permutation of the same set of spins labeling edges of a fixed spin network should be considered equivalent or not, see for example [5]. The counting can be done for both possibilities as explained in [5].

Here we consider the situation in which the counting is only related to different (indistinguishable) configurations. Two configurations which vary by a permutation of spins on the edges of a fixed spin network are considered to be equivalent and are counted only once. We do not worry about other quantum numbers which may be assigned to the punctures such as half integers  $m_I$ , such that  $-j \leq m_I \leq j$  with the projection  $\sum_I m_I = 0$ .

## 2 The counting

The horizon has very large area. First consider the case of equidistant area spectrum, that is  $A = 2 \sum_{i=1}^n (j_i + 1/2)$ .<sup>1</sup> This means that the spectrum becomes equally spaced. In fact the case of equidistant spectrum as a serious candidate for the real spectrum of loop quantum gravity has been considered for example in [11] [12], [13], [14].

In [15] some arguments against the equally spaced spectrum are given due to inconsistency.

But recently it has been shown that we have to consider the equally spaced spectrum as a serious issue [16] as flux-area operator with this property are shown to exist. In [17] the entropy of a black hole is also studied in terms of the equally spaced spectrum. However as the lowest spin which contributes to the area in the case of  $\sqrt{j(j+1)}$  is  $1/2$  we stick to this situation.

In this case equation (3) can be stated as

$$A = 2[n_{1/2}(j_{1/2} + 1/2) + n_1(j_1 + 1/2) + \dots + n_{s_{max}/2}(j_{s_{max}/2} + 1)] \quad (4)$$

which is equivalent to

$$A = n_{1/2}(2j_{1/2} + 1) + n_1(2j_1 + 1) + \dots + n_{s_{max}/2}(2j_{s_{max}/2} + 1) \quad (5)$$

where all the numbers  $n_j$  and  $(2j + 1)$  are integers. Let  $(2j + 1) = m_j$ . For example  $m_{1/2} = 2$  where we assume  $j = 1/2$  to be the lowest value a spin can have. How many configurations satisfy equation (5) exactly? The question translates in calculating how many configurations  $\{n_j\}$  where  $m_j = (2j + 1)$  belonging to the natural numbers exist such that

$$A = n_{1/2}m_{1/2} + n_1m_1 + \dots + n_{s_{max}/2}m_{s_{max}/2} \quad (6)$$

where  $A$  is a natural number.

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<sup>1</sup>Recall we are using units  $4\pi\gamma l_p^2 = 1$

Now we describe how to do the exact counting of different and indistinguishable configurations which satisfy equation (6).

**Definition** A partition of a number  $N$  is  $N = \lambda_1 + \lambda_2 + \dots + \lambda_k$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ . The summands are called the parts of  $N$ . The number of different partitions of  $N$  is denoted  $p(N)$ . [18]

Let us check that any configuration  $\{n_j\} = \{n_{1/2}, n_1, \dots, n_{s_{max}/2}\}$  which satisfies equation (6) is a partition of the number  $A$  according to our definition. Recall that a configuration  $\{n_j\} = \{n_{1/2}, n_1, \dots, n_{s_{max}/2}\}$  of our isolated horizon is a sequence of numbers, where each  $n_j$  only refers to the number of punctures with spin  $j$ . The numbers  $m_j = 2j + 1$  are really of importance since they refer to the dimension of the irreducible representations of  $SU(2)$  of the spin  $j$ .

We therefore think of formula (6) as a partition of the number  $A$  in which its parts are given by the numbers  $m_j$ .

Just note that formula (6) is a sum given by

$$A = (m_{1/2} + m_{1/2} + \dots + m_{1/2}) + (m_1 + m_1 + \dots + m_1) + \dots \\ \dots + (m_{s_{max}/2} + m_{s_{max}/2} + \dots + m_{s_{max}/2})$$

where the first bracketed sum of  $m_{1/2}$  contains  $n_{1/2}$  terms, the second bracketed sum of  $m_1$  contains  $n_1$  terms, and so on, such that the last bracketed sum of  $m_{s_{max}/2}$  contains  $n_{s_{max}/2}$  terms. According to the definition of a partition, the integer area  $A$  needs to be decomposed as  $A = \lambda_1 + \lambda_2 + \dots + \lambda_k$ , with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ ; it is now easy to see that we can take  $\lambda_1 = m_{s_{max}/2}$ ,  $\lambda_2 = m_{s_{max}/2}$ , ...,  $\lambda_{n_{s_{max}/2}} = m_{s_{max}/2}$ , where  $n_{s_{max}/2} \ll k$ . We continue in this way till we get to  $\lambda_{k-n_{1/2}} = m_{1/2}$ , ...,  $\lambda_{k-1} = m_{1/2}$ ,  $\lambda_k = m_{1/2}$ .<sup>2</sup>

Now consider a partition (according to our definition) of the integer number  $A$  which represents the area of our isolated black hole. Since we are considering our minimum allowed spin  $j$  to be  $1/2$ , such that  $m_{1/2} = 2$ , we want to consider partitions of the number  $A$  such that its parts are in terms of natural numbers greater or equal to 2.

Then  $A$  is expressed  $A = \lambda_1 + \lambda_2 + \dots + \lambda_k$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 2$

Then we just think of the integers  $\lambda_j$  as  $m_j$ . Since some of the  $\lambda_j = m_j$  may be repeated it is clear that we will have a decomposition of the number  $A$  as in formula (6).

This implies then that the number of indistinguishable configurations  $\mathcal{N}$  which account for a fixed area  $A$  equals the number of partitions of  $A$  with all parts  $\geq 2$ . An exercise in number theory courses shows that this number is given by  $p(A) - p(A - 1)$ .

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<sup>2</sup>Observe that the partition has all parts  $\geq 2$  since  $m_{1/2} = 2$

It is easy to notice that for the case of equidistant spectrum given by  $A = 2 \sum_{i=1}^n j_i$  where  $j$  is a half integer  $j = m/2$  the number of configurations which account a fixed area  $A$

$$A = n_{1/2}m_{1/2} + n_1m_1 + \dots + n_{s_{max}}m_{s_{max}/2} \quad (7)$$

where the minimum  $m_{1/2} = 1$  is given by the number of partitions of  $A$ , that is  $p(A)$ .

We therefore have the following. If we denote by  $\mathcal{N}_{A_{(j+1/2)}}$  the number of indistinguishable configurations which account a fixed area for the case of equidistant spectrum  $(j+1/2)$  and  $\mathcal{N}_{A_j}$  the number of indistinguishable configurations which account a fixed area for the case of equidistant spectrum  $j$ , we have that

$$\mathcal{N}_{A_j} > \mathcal{N}_{A_{(j+1/2)}} \quad (8)$$

Now for the case of spectrum  $A_j = 2\sqrt{j(j+1)}$  it is known that counting configurations is not restricted to an exact sum but the sum of any configuration gives a number between an interval  $[A - \delta, A + \delta]$ . As the spin  $j = m/2$  for  $m$  integer  $\geq 1$  we can write  $A_j = A_m = \sqrt{m(m+2)} = \sqrt{m^2 + 2m}$ . With this kind of spectrum we can only expect that any configuration will give an irrational number area. One way to go around the problem would be to consider  $A_j \sim (m+1)$  where counting the number of indistinguishable configurations will lead us to the same result we obtained when considering the spectrum  $A_j = (j+1/2)$ .

In Loop Quantum Gravity the number of different indistinguishable configurations  $\mathcal{N}$  which account for a fixed area  $A$  of an isolated horizon is (at least when  $j$  is large) given by  $\mathcal{N} = p(A) - p(A-1)$ .

We could say that the asymptotic behaviour is given by

$$\mathcal{N} \sim \frac{1}{4\sqrt{3}} \left[ \frac{1}{A} \exp(\pi\sqrt{2A/3}) - \frac{1}{(A-1)} \exp(\pi\sqrt{2(A-1)/3}) \right] \quad (9)$$

where in this last formula we have used the asymptotic behaviour of  $p(N)$  known in number theory.

We should say that when counting different indistinguishable configurations, the entropy is asymptotically dominated by the square root of the area. In [5] a counting calculation of indistinguishable configurations shows a similar behaviour to ours.

In [19] a similar formula is given. The difference and contribution in our paper is that we are dealing with the equidistant spectrum and we are also pointing to the number theory methods which we believe are so related to the counting of states of black hole entropy in loop quantum gravity.

For instance in [3] a deep relation to number theory is given. In a future work we plan to deal with the case of the real irrational spectrum of area of loop quantum gravity pointing to more relations to number theory methods.

For instance it is easy to observe that if  $\mathcal{N}_{A_{(j+1/2)}}$  denotes the number of indistinguishable configurations which account a fixed area for the case we treated here of equidistant spectrum  $(j + 1/2)$  and  $\mathcal{N}_{A_{\sqrt{j(j+1)}}}$  the number of indistinguishable configurations which account a fixed area for the case of the original spectrum of loop quantum gravity then

$$\mathcal{N}_{A_{\sqrt{j(j+1)}}} < \mathcal{N}_{A_{(j+1/2)}} \quad (10)$$

which shows that the theory of partitions is essential for the counting.

We are therefore showing that these calculations are profoundly related to number theory. We propose to study in a future work a mathematical rigorous treatment on these relations between counting black hole states in loop quantum gravity and analytic number theory.

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